Hadron spectroscopy

Feng-Kun Guo

Institute of Theoretical Physics, CAS

The Fourth Asia-Europe-Pacific School of High-Energy Physics,
12–25 Sept. 2018, Quy Nhon, Vietnam

Outline

1. **Inroduction**
2. **Hadron resonances discovered since 2003**
   - Open-flavor heavy mesons
   - XYZ states
   - Pentaquark candidates
3. **Theory ideas and applications**
   - Symmetries of QCD: chiral and heavy quark
     - Applications to new heavy hadrons
   - Threshold cusps and triangle singularties
   - Compositeness and hadronic molecules

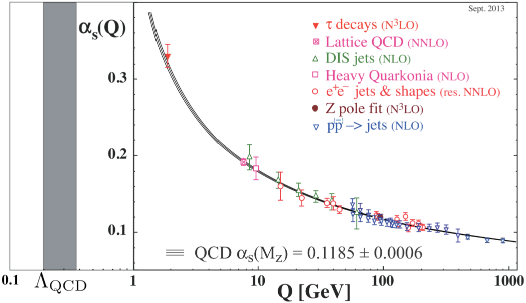
Introduction

Two recent reviews:

- S. L. Olsen, T. Skwarnicki, *Nonstandard heavy mesons and baryons: Experimental evidence*, Rev. Mod. Phys. 90 (2018) 015003 [arXiv:1708.04012]
  experimental facts and interpretations
- F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90 (2018) 015004 [arXiv:1705.00141]
  theoretical formalisms

Two facets of QCD

- Running of the coupling constant $\alpha_s = g_s^2/(4\pi)$



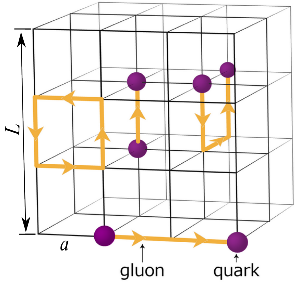
- High energies

lectures by Gavin Salam

  - ☞ asymptotic freedom, perturbative
  - ☞ degrees of freedom: quarks and gluons
- Low energies
  - ☞ nonperturbative, $\Lambda_{\text{QCD}} \sim 300$ MeV $= \mathcal{O}\left(1 \text{ fm}^{-1}\right)$
  - ☞ color confinement, detected particles: mesons and baryons

    $\Rightarrow$ challenge: how do hadrons emerge/how is QCD spectrum organized?

- Lattice QCD: numerical simulation in discretized Euclidean space-time
  - ☞ finite volume (L should be large)
  - ☞ finite lattice spacing (a should be small)
  - ☞ often using $m_{u,d}$ larger than the physical values $\Rightarrow$ chiral extrapolation



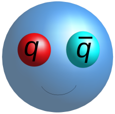
- Phenomenological models, such as quark model, QCD sum rules , ...
- Low-energy EFT:
  mesons and baryons as effective degrees of freedom

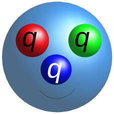
Quarks and hadrons

u	d	s		c	b	t
~2 MeV	~5 MeV	~100 MeV	$\ll \Lambda_{\text{QCD}} \ll$	~1.3 GeV	~4.2 GeV	~173 GeV

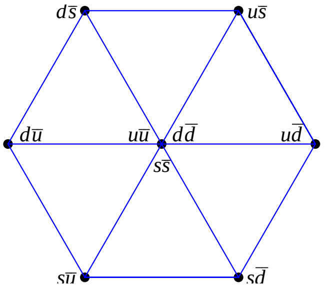
$$\underbrace{u,\ d,\ s}_{\text{Light quarks}} \ll \Lambda_{\text{QCD}} \ll \underbrace{c,\ b,\ t}_{\text{Heavy quarks}}$$

☞ Mesons and baryons in quark model





Light meson SU(3) [u, d, s] multiplets (octet + singlet):



- Vector mesons

meson	quark content	mass (MeV)
ρ^+/ρ^-	$u\bar{d}\,/\,d\bar{u}$	775
ρ^0	$(u\bar{u}-d\bar{d})/\sqrt{2}$	775
K^{*+}/K^{*-}	$u\bar{s}\,/\,s\bar{u}$	892
$K^{*0}/\bar{K}^{*0}$	$d\bar{s}\,/\,s\bar{d}$	896
ω	$(u\bar{u}+d\bar{d})/\sqrt{2}$	783
ϕ	$s\bar{s}$	1019

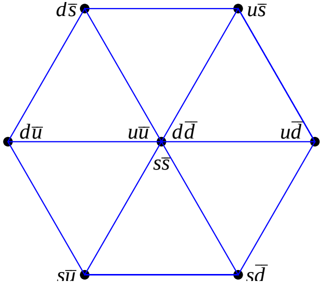
☞ approximate SU(3) symmetry

$$m_\rho \simeq m_\omega, \qquad m_\phi - m_{K^*} \simeq m_{K^*} - m_\rho$$

☞ very good isospin SU(2) symmetry

$$m_{\rho^0} - m_{\rho^\pm} = (-0.7 \pm 0.8)\ \text{MeV}, \quad m_{K^{*0}} - m_{K^{*\pm}} = (6.7 \pm 1.2)\ \text{MeV}$$

Light meson SU(3) [u, d, s] multiplets (octet + singlet):



- Pseudoscalar mesons

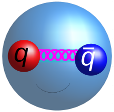
meson	quark content	mass (MeV)
π^+/π^-	$u\bar{d}\,/\,d\bar{u}$	140
π^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	135
K^+/K^-	$u\bar{s}\,/\,s\bar{u}$	494
$K^0/\bar{K}^0$	$d\bar{s}\,/\,s\bar{d}$	498
η	$\sim (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$	548
η'	$\sim (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$	958

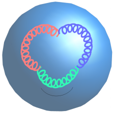
☞ very good isospin SU(2) symmetry

$$m_{\pi^\pm} - m_{\pi^0} = (4.5936 \pm 0.0005)\ \text{MeV}, \quad m_{K^0} - m_{K^\pm} = (3.937 \pm 0.028)\ \text{MeV}$$

☞ Q: Why are the pions so light?

- Quark model notation:
  any hadron resonances beyond picture of $q\bar{q}$ for a meson and qqq for a baryon
  - ☞ Gluoinc excitations: hybrids and glueballs

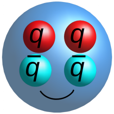




  - ☞ Multiquark states

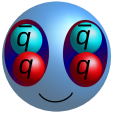
A Schematic Model of Baryons and Mesons

M.Gell-Mann, Phys.Lett.**8**(1964)214-215



We then refer to the members $u^{\frac{2}{3}}$, $d^{-\frac{1}{3}}$, and $s^{-\frac{1}{3}}$ of the triplet as "quarks" 6) q and the members of the anti-triplet as anti-quarks $\bar{q}$. Baryons can now be constructed from quarks by using the combinations $(q\,q\,q)$, $(q\,q\,q\,q\,\bar{q})$, etc., while mesons are made out of $(q\,\bar{q})$, $(q\,q\,\bar{q}\,\bar{q})$, etc. It is assuming that the lowest baryon configuration $(q\,q\,q)$ gives just the representations **1**, **8**, and **10** that have been observed, while the lowest meson configuration $(q\,\bar{q})$ similarly gives just **1** and **8**.

- Hadronic molecules:
  bound states of two or more hadrons,
  analogues of nuclei



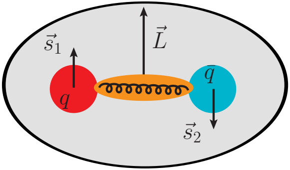
- J^{PC} of regular $q\bar{q}$ mesons

  $P = (-1)^{L+1}$

  $C = (-1)^{L+S}$ for flacor-neutral mesons

  L: orbital angular momentum

  $S = (0, 1)$: total spin of q and $\bar{q}$



☞ For $S = 0$, the meson spin $J = L$, one has $P = (-1)^{J+1}$ and $C = (-1)^J$. Hence,

$$J^{PC} = \text{even}^{-+} \text{ and odd}^{+-}$$

☞ For $S = 1$, one has $P = C = (-1)^{L+1}$. Hence,

$$J^{PC} = 1^{--}, \{0, 1, 2\}^{++}, \{1, 2, 3\}^{--}, \ldots$$

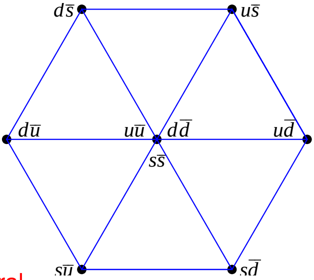
- Exotic J^{PC} for mesons:

$$\boxed{J^{PC} = 0^{--}, \text{even}^{+-} \text{ and odd}^{-+}}$$

Additive quantum numbers

Some trivial facts about additive quantum numbers of regular mesons

- Light-flavor mesons (here S =strangeness)
  - Nonstrange mesons: $S = 0$, $I = 0, 1$
  - Strange mesons: $S = \pm 1$, $I = \frac{1}{2}$
- Open-flavor heavy mesons
  - $Q\bar{q}(q = u, d)$: $S = 0$, $I = 1/2$
  - $Q\bar{s}$: $S = 1$, $I = 0$
- Heavy quarkonia ($Q\bar{Q}$): $S = 0, I = 0$, neutral



Charge, isospin, strangeness etc. which cannot be achieved in the $q\bar{q}$ and qqq scheme would be a smoking gun for an exotic nature

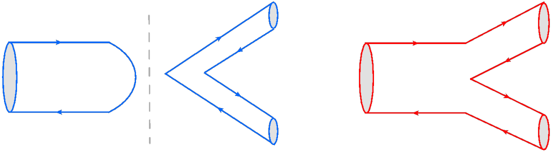
more subtlety later...

- SU(3) flavor symmetry is usually satisfied to 30%

$\boxed{Example}$

$$\frac{\Gamma(K^{*+})}{\Gamma(\rho^+)} = \frac{51\text{ MeV}}{149\text{ MeV}} = 0.34 \text{ [exp]}, \qquad \frac{3}{4}\left(\frac{M_\rho}{M_{K^*}}\right)^2\left(\frac{q_{K\pi}}{q_{\pi\pi}}\right)^3 = 0.29 \text{ [SU(3)]}$$

- Okubo–Zweig–Iizuka (OZI) rule:



drawing only quark lines, the disconnected diagrams are strongly suppressed relative to the connected ones

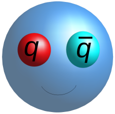
$\boxed{Example}$

$\psi(3770)$: ∼40 MeV above the $D\bar{D}$ threshold

$$\mathcal{B}(D\bar{D}) = (93^{+8}_{-9})\% \gg \mathcal{B}(\text{sum of any other modes})$$

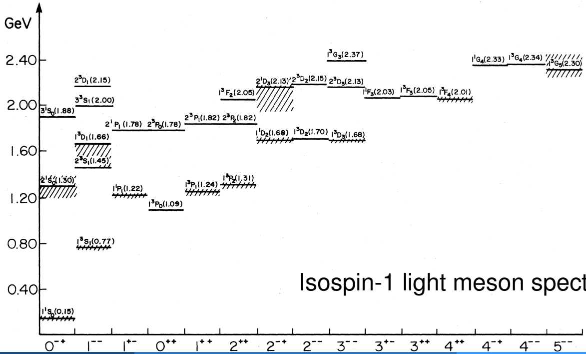
Godfrey–Isgur quark model

Mesons in a Relativized Quark Model with Chromodynamics
S. Godfrey, Nathan Isgur (Toronto U.). 1985. 43 pp.
Published in **Phys.Rev. D32 (1985) 189-231**
DOI: 10.1103/PhysRevD.32.189



$$\left(\sqrt{m_1^2+p^2}+\sqrt{m_2^2+p^2}+V\right)|\Psi\rangle = E|\Psi\rangle$$

Potential V: One-gluon exchange + linear confinement + relativistic effects



Isospin-1 light meson spectroscopy

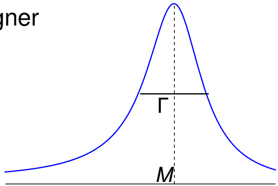
New discoveries since 2003

Many new hadron resonances observed in experiments since 2003

- Inactive: BaBar, Belle, CDF, CLEO-c, D0, ...
- Running: Belle-II, BESIII, COMPASS, LHCb, ...
- Under construction/discussion: PANDA, EIC, EicC, ...

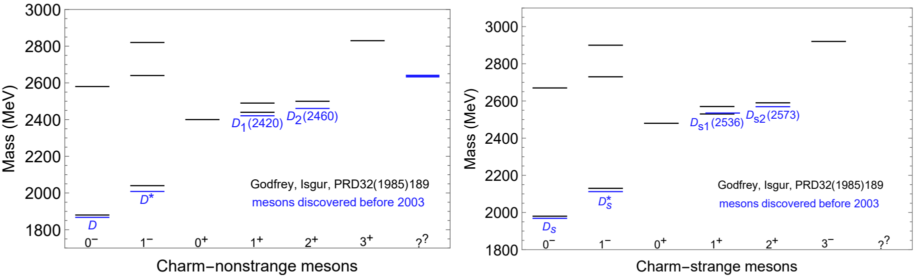
Common strategy: search for peaks, fit with Breit–Wigner

$$\propto \frac{1}{(s-M^2)^2 + s\,\Gamma^2(s)}$$



Lots of mysteries right now ...

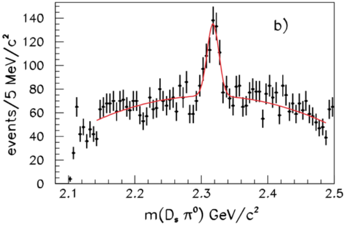
Open-flavor heavy mesons

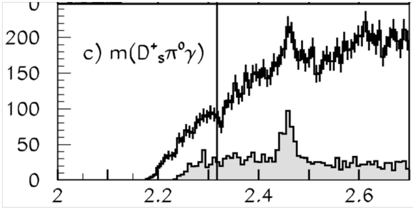


Most quark-model predicted states were still missing before 2003

Discoveries in 2003 (both Belle and BaBar started data taking in 1999):

- $D^*_{s0}(2317)$: discovered in $e^+e^- \to D_s^+\pi^0 X$ BaBar, PRL90(2003)242001 [hep-ex/0304021]





$J^P = 0^+$, $M = (2317.7 \pm 0.6)$ MeV, $\Gamma < 3.8$ MeV

$I = 0$, $\to D_s\pi^0$: breaks isospin symmetry

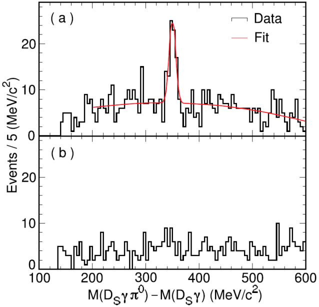
- $D_{s1}(2460)$: discovered in $e^+e^- \to D_s^{*+}\pi^0 X$

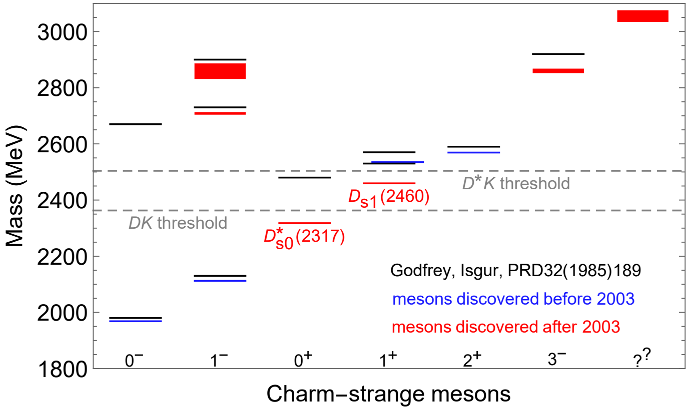
CLEO, PRD68(2003)032002 [hep-ex/0305100]

$J^P = 1^+$, $M = (2459.5 \pm 0.6)$ MeV, $\Gamma < 3.5$ MeV

$I = 0$, $\to D_s^*\pi^0$: breaks isospin symmetry

other decays: $D_s^+\gamma$, $D_s^+\pi^+\pi^-$, $D^*_{s0}(2317)\gamma$





$D_{s0}^*(2317)$ and $D_{s1}(2460)$: the first established new hadrons

- Puzzle 1: Why are $D_{s0}^*(2317)$ and $D_{s1}(2460)$ so light?
- Puzzle 2: Why $\underbrace{M_{D_{s1}(2460)} - M_{D_{s0}^*(2317)}}_{=(141.8\pm0.8)\text{ MeV}} \simeq \underbrace{M_{D^{*\pm}} - M_{D^\pm}}_{=(140.67\pm0.08)\text{ MeV}}$?

Charm-nonstrange mesons (1)

Obervations of charm-nonstrange excited mesons in 2003

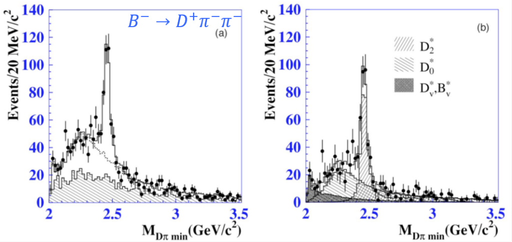
$B^- \to D^{(*)+}\pi^-\pi^-$ Belle, PRD69(2004)112002 [hep-ex/0307021]

- $D_0^*(2400)$: $J^P = 0^+$

  $\Gamma = (267 \pm 40)$ MeV

  Mass (MeV):

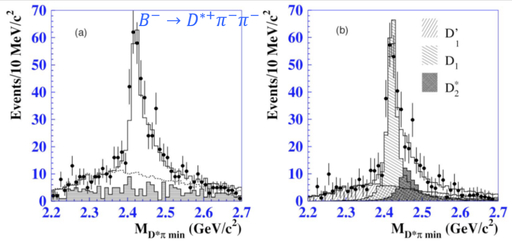
2318 ± 29	PDG18	
2297 ± 22	BaBar	B decays
2308 ± 36	Belle	B decays
2401 ± 41	FOCUS	γA
2360 ± 34	LHCb	B decays

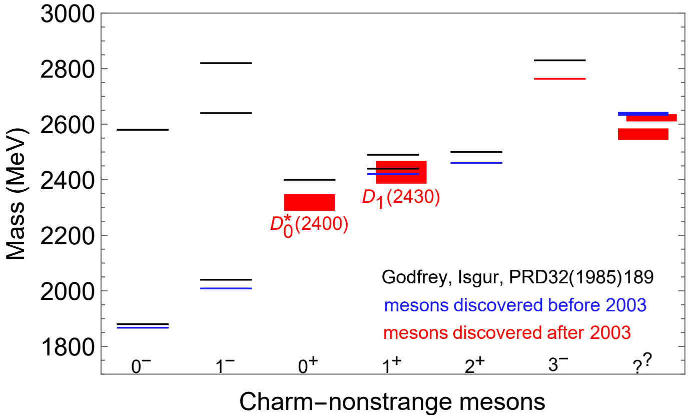


- $D_1(2430)$: $J^P = 1^+$

  $\Gamma = 384^{+130}_{-110}$ MeV

  $M = (2427 \pm 36)$ MeV

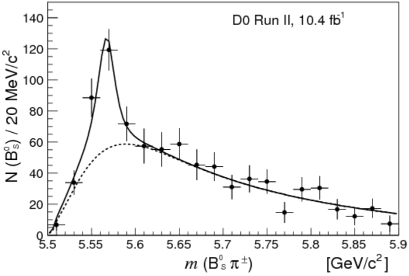




- Puzzle 3: Why $M_{D_0^*(2400)} \gtrsim M_{D_{s0}^*(2317)}$ and $M_{D_1(2430)} \sim M_{D_{s1}(2460)}$?

Most exotic and newest observation: $X(5568)$

- $X(5568)$ by D0 Collaboration ($p\bar{p}$ collisions) PRL117(2016)022003; PRD97(2018)092004



$$M = \left(5567.8 \pm 2.9^{+0.9}_{-1.9}\right) \text{ MeV}$$

$$\Gamma = \left(21.9 \pm 6.4^{+5.0}_{-2.5}\right) \text{ MeV}$$

- Observed in $B_s^{(*)0}\pi^+$, sizeable width $\Rightarrow I = 1$:

  minimal quark contents is $\bar{b}s\bar{d}u$!

- a favorite mulqituark candidate:

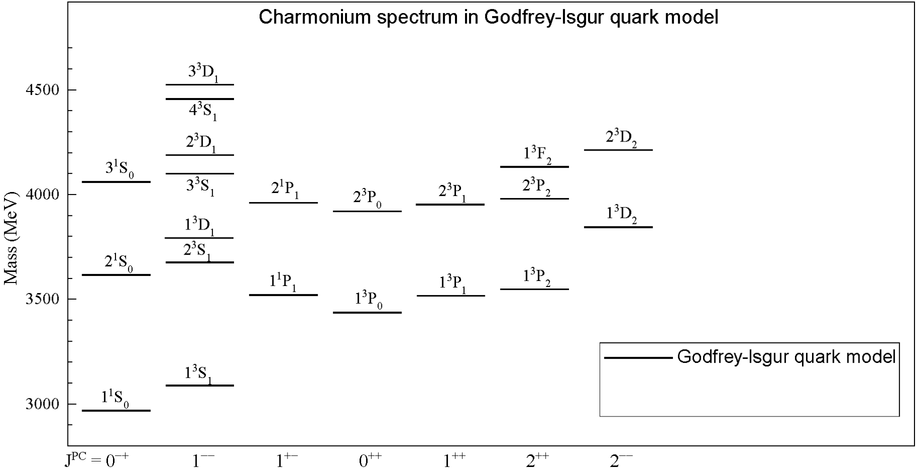
  explicitly flavor exotic, minimal number of quarks ≥ 4

Estimate of isospin breaking decay width:

$$\Gamma_{\not{I}} \sim \left(\frac{\left(\frac{m_d - m_u}{\Lambda_{\rm QCD}}\right)^2}{\alpha^2} \right) \times \mathcal{O}(100 \text{ MeV})$$

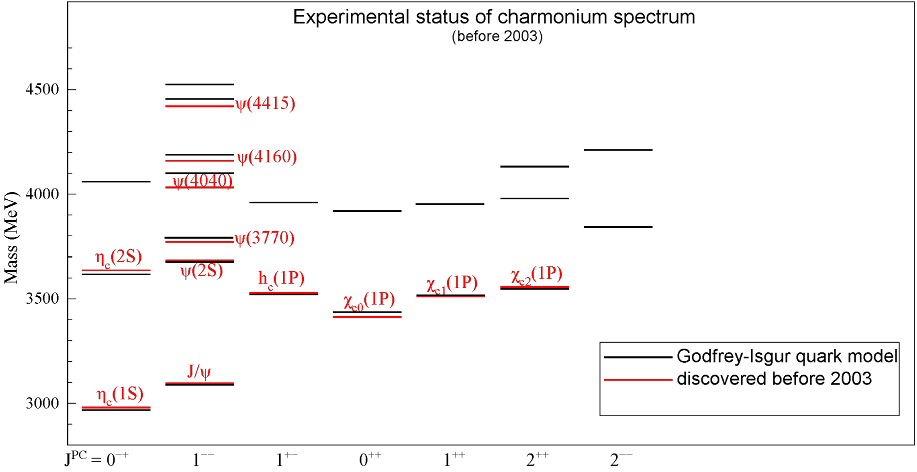
$$= \mathcal{O}(10 \text{ keV})$$

XYZ states



Charmonium spectrum in Godfrey-Isgur quark model
Mass (MeV)
4500
4000
3500
3000
3^3D_1
4^3S_1
2^3D_1
3^3S_1
1^3D_1
2^3S_1
1^3S_1
3^1S_0
2^1S_0
1^1S_0
2^1P_1
1^1P_1
2^3P_0
1^3P_0
2^3P_1
1^3P_1
1^3F_2
2^3P_2
1^3P_2
2^3D_2
1^3D_2
Godfrey-Isgur quark model
$J^{PC} = 0^{-+}$
1^{--}
1^{+-}
0^{++}
1^{++}
2^{++}
2^{--}

XYZ states



Experimental status of charmonium spectrum
(before 2003)
Mass (MeV)
4500
4000
3500
3000
ψ(4415)
ψ(4160)
ψ(4040)
ψ(3770)
ψ(2S)
J/ψ
η_c(2S)
η_c(1S)
h_c(1P)
χ_c0(1P)
χ_c1(1P)
χ_c2(1P)
Godfrey-Isgur quark model
discovered before 2003
J^PC = 0^-+
1^--
1^+-
0^++
1^++
2^++
2^--

Naming convention

For states with properties in conflict with naive quark model (normally):

- X: $I = 0$, J^{PC} other than 1^{--} or unknown
- Y: $I = 0$, $J^{PC} = 1^{--}$
- Z: $I = 1$

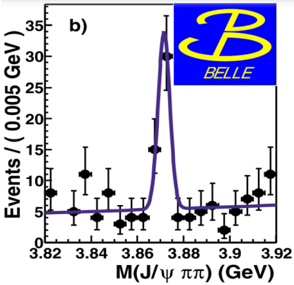
PDG2018 naming scheme:

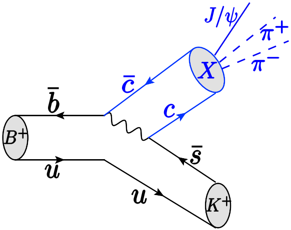
$J^{PC} =$ / Minimal quark content	0^{-+}, 2^{-+}, $\vdots$	1^{+-}, 3^{+-}, $\vdots$	1^{--}, 2^{--}, $\vdots$	0^{++}, 1^{++}, $\vdots$
$u\bar{d}, u\bar{u} - d\bar{d}, d\bar{u}$ $(I = 1)$	π	b	ρ	a
$d\bar{d} + u\bar{u}$ and/or $s\bar{s}$ $(I = 0)$	η, η'	h, h'	ω, ϕ	f, f'
$c\bar{c}$	η_c	h_c	$\psi^\dagger$	χ_c
$b\bar{b}$	η_b	h_b	Υ	χ_b
$I = 1$ with $c\bar{c}$	(Π_c)	Z_c	R_c	(W_c)
$I = 1$ with $b\bar{b}$	(Π_b)	Z_b	(R_b)	(W_b)

†The J/ψ remains the J/ψ.

"Young man, if I could remember the names of these particles, I would have been a botanist." — *Enrico Fermi*

Belle, PRL91(2003)262001 [hep-ex/0309032]





- The beginning of the XYZ story, discovered in $B^\pm \to K^\pm J/\psi\pi\pi$

  $M_X = (3871.69 \pm 0.17)$ MeV
- $\Gamma < 1.2$ MeV Belle, PRD84(2011)052004
- Confirmed in many experiments: Belle, BaBar, BESIII, CDF, CMS, D0, LHCb, ...
- 10 years later, $J^{PC} = 1^{++}$

  LHCb, PRL110(2013)222001

  $\Rightarrow$ S-wave coupling to $D\bar{D}^*$

Mysterious properties:

- Mass coincides with the $D^0\bar{D}^{*0}$ threshold:

  $M_{D^0} + M_{D^{*0}} - M_X = (0.00 \pm 0.18)$ MeV

$X(3872)$ (2)

Mysterious properties (cont.):

- Large coupling to $D^0\bar{D}^{*0}$:

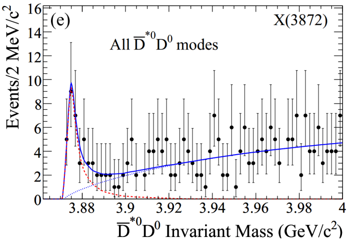
  $\mathcal{B}(X\to D^0\bar{D}^{*0}) > 30\%$ Belle, PRD81(2010)031103

  $\mathcal{B}(X\to D^0\bar{D}^0\pi^0) > 40\%$ Belle, PRL97(2006)162002

- No isospin partner observed $\Rightarrow I=0$

  but, large isospin breaking:

$$\frac{\mathcal{B}(X\to \omega J/\psi)}{\mathcal{B}(X\to \pi^+\pi^- J/\psi)} = 0.8\pm 0.3$$



BaBar, PRD77(2008)011102

$$C(X)=+, C(J/\psi)=- \quad \Rightarrow C(\pi^+\pi^-)=- \quad \Rightarrow I(\pi^+\pi^-)=1$$

- Radiative decays:

$$\frac{\mathcal{B}(X\to \gamma\psi')}{\mathcal{B}(X\to \gamma J/\psi)} = 2.6\pm 0.6$$

PDG18 average of BaBar(2009) and LHCb(2014) measurements

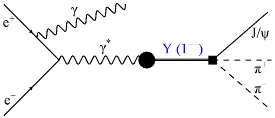
Exercise:

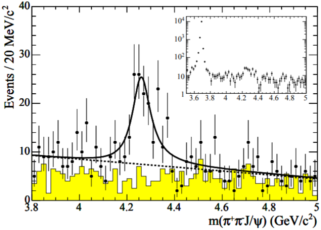
1) Why is the isospin of the negative C-parity $\pi^+\pi^-$ system equal to 1?

2) Is $\Upsilon\pi^+\pi^-$ a good choice of final states for the search of X_b, the $J^{PC}=1^{++}$ bottom analogue of the $X(3872)$?

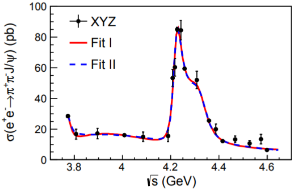
$Y(4260)$

- Discovered by BaBar in 2005 PRL95(2005)142001

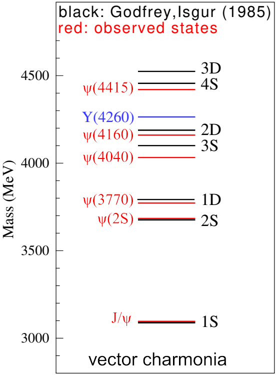




$J^{PC} = 1^{--}$, confirmed by Belle, CLEO, BESIII



BESIII, PRL118(2017)092001

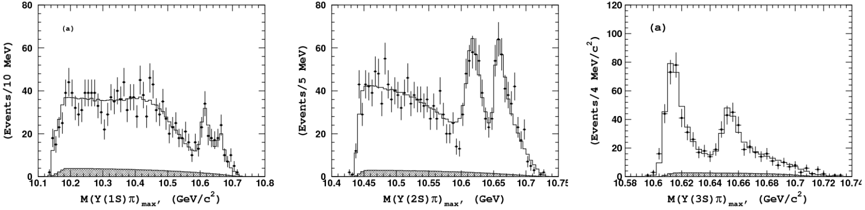


- $M = (4230 \pm 8)$ MeV, $\Gamma = (55 \pm 19)$ MeV PDG2018
- Puzzles:
  - ☞ no slot in quark model
  - ☞ well above $D\bar{D}$ threshold, but not seen in $D\bar{D}$ (recall the OZI rule)

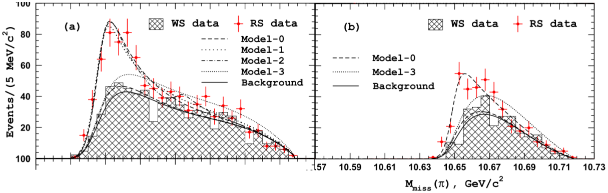
$Z_c^\pm$ and $Z_b^\pm$ (1)

- $Z_c^\pm$, $Z_b^\pm$: charged structures in heavy quarkonium mass region, excellent tetraquark candidates: $Q\bar{Q}\bar{d}u, Q\bar{Q}\bar{u}d$
- $Z_b(10610)^\pm$ and $Z_b(10650)^\pm$: Belle, arXiv:1105.4583; PRL108(2012)122001

  observed in $\Upsilon(10860) \to \pi^\mp[\pi^\pm\Upsilon(1S,2S,3S)/h_b(1P,2P)]$



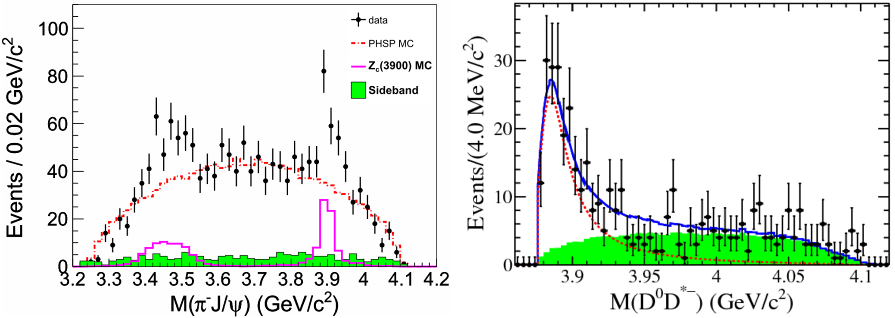
  also in $\Upsilon(10860) \to \pi^\mp[B^{(*)}\bar{B}^*]^\pm$ Belle, arXiv:1209.6450; PRL116(2016)212001



- $Z_b(10610)^\pm$ and $Z_b(10650)^\pm$ very close to $B\bar{B}^*$ and $B^*\bar{B}^*$ thresholds

$Z_c^\pm$ and $Z_b^\pm$ (2)

- $Z_c(3900)^\pm$: structure around 3.9 GeV seen in $J/\psi\pi^\pm$ by BESIII and Belle in $Y(4260) \to J/\psi\pi^+\pi^-$, BESIII, PRL110(2013)252001; Belle, PRL110(2013)252002
  and in $D\bar{D}^*$ by BESIII in $Y(4260) \to \pi^\pm(D\bar{D}^*)^\mp$ BESIII, PRD92(2015)092006

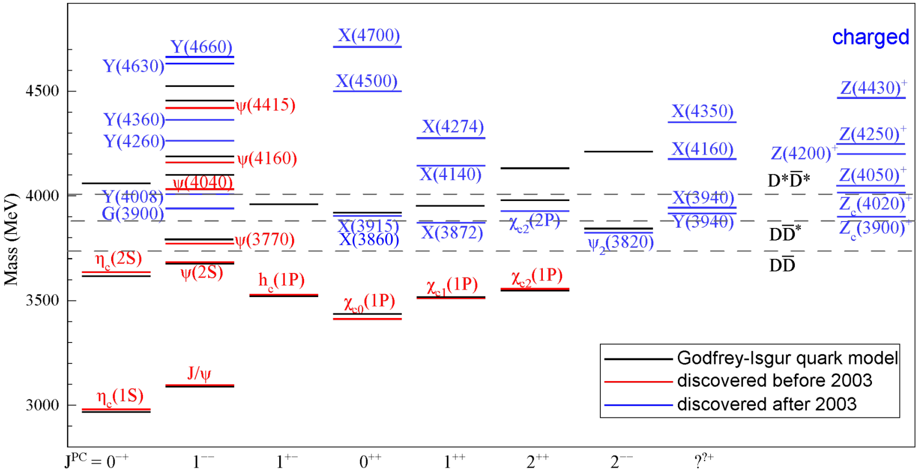


- $Z_c(4020)^\pm$ observed in $h_c\pi^\pm$ and $(\bar{D}^*D^*)^\pm$ distributions

  BESIII, PRL111(2013)242001; PRL112(2014)132001

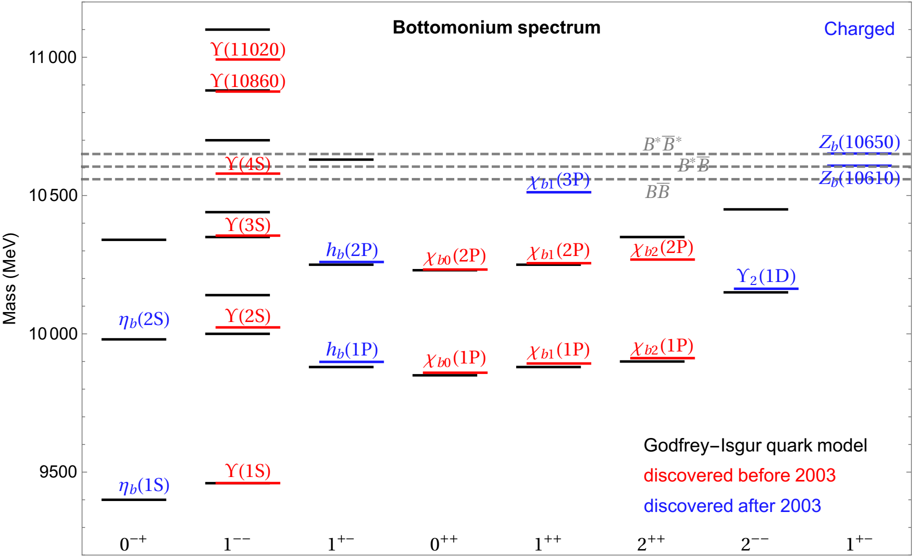
- $Z_c(3900)^\pm$ and $Z_c(4020)^\pm$ very close to $D\bar{D}^*$ and $D^*\bar{D}^*$ thresholds

Charmonium spectrum: current status



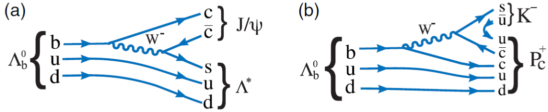
Note: J^{PC} of $X(3915)$ might also be 2^{++}

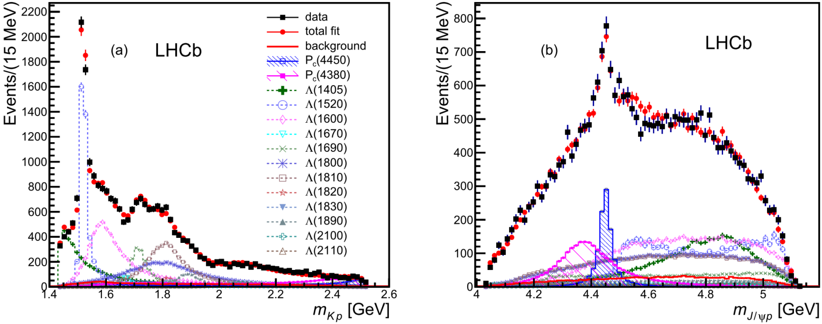
Bottomonium spectrum: current status



Pentaquark candidates

LHCb, PRL115(2015)072001 [arXiv:1507.03414]





Two Breit–Wigner resonances needed:

$$M_1 = (4380 \pm 8 \pm 29)\ \text{MeV}, \qquad \Gamma_1 = (205 \pm 18 \pm 86)\ \text{MeV},$$

$$M_2 = (4449.8 \pm 1.7 \pm 2.5)\ \text{MeV}, \qquad \Gamma_2 = (39 \pm 5 \pm 19)\ \text{MeV}.$$

- In $J/\psi\, p$ invariant mass distribution, with hidden charm
  $\Rightarrow$ pentaquarks if they are hadron resonances
- Quantum numbers not fully determined, for ($P_c(4380), P_c(4450)$):
  $(3/2^-, 5/2^+)$, $(3/2^+, 5/2^-)$, $(5/2^+, 3/2^-)$, $\ldots$

  From a reanalysis using an extended Λ^* model: N. Jurik, CERN-THESIS-2016-086

		$P_c(4380)$		$P_c(4450)$	
$J^p(4380, 4450)$	$(\sqrt{\Delta(-2\ln\mathcal{L})})^2$	M_0	Γ_0	M_0	Γ_0
	$(3/2-, 5/2^+)$ solution				
$3/2^-, 5/2^+$	$--$	4359	151	4450.1	49
	Δ from $(3/2-, 5/2^+)$ solution				
$5/2^+, 3/2^-$	-3.6^2	10	-7	-1.6	-6
$5/2^-, 3/2^+$	-2.7^2	-4	-9	-3.6	-2
$3/2^-, 5/2^+$	–	–	–	–	–

- Early prediction:
  Prediction of narrow N^* and Λ^* resonances with hidden charm above 4 GeV,
  J.-J. Wu, R. Molina, E. Oset, B.-S. Zou, PRL105(2010)232001

Recent reviews on new hadrons

- H.-X. Chen et al., *The hidden-charm pentaquark and tetraquark states*, Phys. Rept. 639 (2016) 1 [arXiv:1601.02092]
- A. Hosaka et al., *Exotic hadrons with heavy flavors — X, Y, Z and related states*, Prog. Theor. Exp. Phys. 2016, 062C01 [arXiv:1603.09229]
- R. F. Lebed, R. E. Mitchell, E. Swanson, *Heavy-quark QCD exotica*, Prog. Part. Nucl. Phys. 93 (2017) 143, arXiv:1610.04528 [hep-ph]
- A. Esposito, A. Pilloni, A. D. Polosa, *Multiquark resonances*, Phys. Rept. 668 (2017) 1 [arXiv:1611.07920]
- F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90 (2018) 015004 [arXiv:1705.00141]
- S. L. Olsen, T. Skwarnicki, *Nonstandard heavy mesons and baryons: Experimental evidence*, Rev. Mod. Phys. 90 (2018) 015003 [arXiv:1708.04012]
- M. Karliner, J. L. Rosner, T. Skwarnicki, *Multiquark states*, Ann. Rev. Nucl. Part. Sci. 68 (2018) 17 [arXiv:1711.10626]
- C.-Z. Yuan, *The XYZ states revisited*, Int. J. Mod. Phys. A 33 (2018) 1830018 [arXiv:1808.01570]

Symmetries of QCD: chiral and heavy quark

Useful monographs:

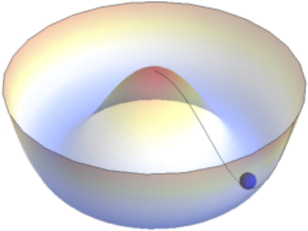
- H. Georgi, *Weak Interactions and Modern Particle Physics* (2009)
- J.F. Donoghue, E. Golowich, B.R. Holstein, *Dynamics of the Standard Model* (1992)
- S. Scherer, M.R. Schindler, *A Primer for Chiral Perturbation Theory* (2012)
- A.V. Manohar, M.B. Wise, *Heavy Quark Physics* (2000)

- Different quark flavors:

u	d	s		c	b	t
~2 MeV	~5 MeV	~100 MeV	$\ll \Lambda_{\text{QCD}} \ll$	~1.3 GeV	~4.2 GeV	~173 GeV

Light quarks (u, d, s) — Heavy quarks (c, b, t)

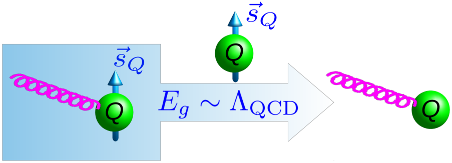
☞ Spontaneously broken chiral symmetry: π, K and η as the pseudo-Goldstone bosons



☞ Heavy quark spin symmetry

☞ Heavy quark flavor symmetry

☞ Heavy antiquark-diquark symmetry



Chiral symmetry in a nut shell

Chiral symmetry (1)

- QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - \left(\bar{q}_L \mathcal{M} q_R + \bar{q}_R \mathcal{M} q_L\right) + \ldots$$

$$q = \tfrac{1}{2}(1-\gamma_5)q + \tfrac{1}{2}(1+\gamma_5)q \equiv P_L q + P_R q \;=\; q_L + q_R$$

- For $m_{u,d,s} = 0$, invariant under U$(3)_L \times$U(3)$_R$ transformations:

$$\mathcal{L}^0_{\rm QCD}(G_{\mu\nu},\, q',\, D_\mu q') = \mathcal{L}^0_{\rm QCD}(G_{\mu\nu},\, q,\, D_\mu q)$$

$$q' = R\, P_R q + L\, P_L q = R\, q_R + L\, q_L$$

$$R \in \mathrm{U}(3)_R, \quad L \in \mathrm{U}(3)_L$$

- Parity: $q(t,\vec{x}) \xrightarrow{P} \gamma^0 q(t,-\vec{x})$

$\Rightarrow$ $q_R(t,\vec{x}) \xrightarrow{P} P_R \gamma^0 q(t,-\vec{x}) = \gamma^0 P_L q(t,-\vec{x}) = \gamma^0 q_L(t,-\vec{x})$

$q_L(t,\vec{x}) \xrightarrow{P} \gamma^0 q_R(t,-\vec{x})$

- $\mathrm{U}(3)_L \times \mathrm{U}(3)_R = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$

$$q_{L/R} \Rightarrow \text{\L}_{/R}\, q_{L/R} = e^{-i\alpha^a_{L/R}T^a} e^{-i\alpha_{L/R}} q_{L/R}.$$

$$q = q_L + q_R = \frac{1-\gamma_5}{2} q + \frac{1+\gamma_5}{2} q$$

$$\longrightarrow \frac{1}{2}\left(e^{-i\alpha^a_L T^a} e^{-i\alpha_L} + e^{-i\alpha^a_R T^a} e^{-i\alpha_R}\right) q + \frac{\gamma_5}{2}\left(-e^{-i\alpha^a_L T^a} e^{-i\alpha_L} + e^{-i\alpha^a_R T^a} e^{-i\alpha_R}\right) q$$

$$= \frac{1}{2}\left(2 - i\alpha^a_L T^a - i\alpha_L - i\alpha^a_R T^a - i\alpha_R\right) q + \frac{\gamma_5}{2}\left(i\alpha^a_L T^a + i\alpha_L - i\alpha^a_R T^a - i\alpha_R\right) q + \ldots$$

$$= \Big(1 - i\underbrace{\alpha^a_V}_{\frac{1}{2}(\alpha^a_L+\alpha^a_R)} T^a - i\underbrace{\alpha_V}_{\frac{1}{2}(\alpha_L+\alpha_R)}\Big) q + \Big(-i\underbrace{\alpha^a_A}_{\frac{1}{2}(\alpha^a_R-\alpha^a_L)} T^a - i\underbrace{\alpha_A}_{\frac{1}{2}(\alpha_R-\alpha_L)}\Big)\gamma_5 q + \ldots$$

$$= \underbrace{e^{-i\alpha^a_V T^a}}_{SU(N)_V} \underbrace{e^{-i\alpha_V}}_{U(1)_V} \underbrace{e^{-i\alpha^a_A T^a\gamma_5}}_{SU(N)_A} \underbrace{e^{-i\alpha_A\gamma_5}}_{U(1)_A} q$$

- $\mathrm{U}(3)_L \times \mathrm{U}(3)_R = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \underbrace{\mathrm{U}(1)_V}_{\text{baryon number cons.}} \times \underbrace{\mathrm{U}(1)_A}_{\text{broken by quantum anomaly}}$

Note: “SU(3)$_A$” not a group

Chiral symmetry (3): Wigner–Weyl v.s. Nambu–Goldstone

- Noether's theorem: continuous symmetry $\Rightarrow$ conserved currents

  Let Q^a be symmetry charges: $Q^a = \int d^3\vec{x}\, J^{a,0}(t,\vec{x}), \quad \partial_\mu J^{a,\mu} = 0$
- Q^a is the symmetry generator: $g = e^{i\alpha^a Q^a}$, H: Hamiltonian, thus

$$gHg^{-1} = H \;\Rightarrow\; [Q^a, H] = 0,$$

$$[Q^a, H]|0\rangle = Q^a \underbrace{H|0\rangle}_{=0} - HQ^a|0\rangle = 0$$

- Wigner–Weyl mode: $Q^a|0\rangle = 0$ or equivalently $g|0\rangle = |0\rangle$

  degeneracy in mass spectrum
- Nambu–Goldstone mode: $g|0\rangle \neq |0\rangle$, spontaneously broken (hidden)

  $Q^a|0\rangle \neq |0\rangle$: new states degenerate with vacuum, massless Goldstone bosons
  - ☞ spontaneously broken continuous global symmetry $\Rightarrow$ massless GBs
  - ☞ the same quantum numbers as $Q^a|0\rangle \Rightarrow$ spinless
  - ☞ $\#$(GBs) $= \#$(broken generators)

A. Zee: "*If you want to show off your mastery of mathematical jargon you can say that the Nambu–Goldstone bosons live in the coset space G/H.*"

Chiral symmetry (4)

- $PQ_A^a P^{-1} = -Q_A^a$, if $G = \text{SU}(N_f)_L \times \text{SU}(N_f)_R$ realized in Wigner-Weyl mode $\Rightarrow$ parity doubling: hadrons have degenerate partners with opposite parity, but

$$m_{\text{Nucleon},\ P=+} = 939\ \text{MeV} \ll m_{N^*(1535),\ P=-} = 1535\ \text{MeV},$$

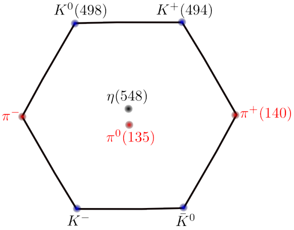
$$m_{\pi,\ P=-} = 139\ \text{MeV} \ll m_{a_0(980),\ P=+} = 980\ \text{MeV}$$

- vacuum invariant under $H = \text{SU}(N_f)_V$: $\quad Q_V^a|0\rangle = 0, \quad Q_A^a|0\rangle \neq 0$
- SSB $\Rightarrow$ massless pseudoscalar Goldstone bosons

  $\#(\text{GBs}) = \dim(G) - \dim(H) = N_f^2 - 1$

  for $N_f = 3$, 8 GBs: $\quad \pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

  for $N_f = 2$, 3 GBs: $\quad \pi^\pm, \pi^0$



- Pions get a small mass due to explicit symmetry breaking by tiny $m_{u,d}$ (a few MeV)

  peudo-Goldstone bosons, Gell-Mann–Oakes–Renner: $M_\pi^2 \propto (m_u + m_d)$

  $M_\pi \ll M_{\text{other hadron}}, \quad$ also, $m_s \gg m_{u,d} \;\Rightarrow\; M_K \gg M_\pi$
- Mechanism for $\text{SU}(N_f)_L \times \text{SU}(N_f)_R \to \text{SU}(N_f)_V$ in QCD not understood

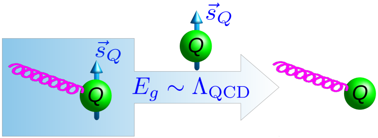
Heavy quark symmetries

- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer Λ_{QCD}

☞ heavy quark spin symmetry (HQSS):

chromomag. interaction $\propto \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{m_Q}$

spin of the heavy quark decouples



Let total angular momentum $\boldsymbol{J} = \boldsymbol{s}_Q + \boldsymbol{s}_\ell$,

$\boldsymbol{s}_Q$: heavy quark spin,

$\boldsymbol{s}_\ell$: spin of the light degrees of freedom (including orbital angular momentum)

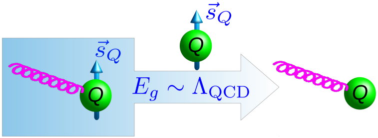
- ⋆ angular momentum conservation $\overset{m_Q\to+\infty}{\Longrightarrow}$ s_ℓ is conserved
- ⋆ spin multiplets:

  for singly-heavy mesons, e.g., $\{D, D^*\}$ with $s_\ell^P = \frac{1}{2}^-$,

  $M_{D^*} - M_D \simeq 140$ MeV, $\qquad M_{B^*} - M_B \simeq 46$ MeV

  for $Q\bar{Q}$, e.g., $\{\eta_c, J/\psi\}$, $\{\chi_{c0}, \chi_{c1}, \chi_{c2}, h_c\}$, $\{\eta_b, \Upsilon\}$

- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer Λ_{QCD}
  - ☞ heavy quark flavor symmetry (HQFS):
    for any hadron containing one heavy quark:
    velocity remains unchanged in the limit $m_Q \to \infty$:



$$\Delta v = \frac{\Delta p}{m_Q} = \frac{\Lambda_{\text{QCD}}}{m_Q}$$

$\Rightarrow$ heavy quark is like a static color triplet source, m_Q is irrelevant

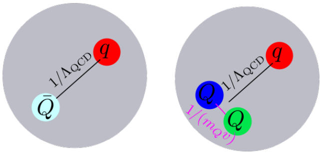
  - ☞ heavy anti-quark–diquark symmetry

M. Savage, M. Wise, PLB248(1990)177

if $m_Q v \gg \Lambda_{\text{QCD}}$,

the diquark serves as a point-like color-$\bar{3}$ source, like a heavy anti-quark.

It relates doubly-heavy baryons to anti-heavy mesons



Intermediate summary

- Many new hadrons observed (in particular in the charm sector), lots of mysteries
- Symmetries of QCD:
  - spontaneously broken chiral symmetry for light flavors
  - heavy quark spin and flavor symmetry for heavy flavors

$\Rightarrow$ next, applications of symmetries to the new hadrons

HQS for open-flavor heavy hadrons

Examples of HQSS phenomenology:

- Widths of the two D_1 ($J^P = 1^+$) mesons
  - ☞ $\Gamma[D_1(2420)] = (27.4 \pm 2.5)$ MeV $\ll$ $\Gamma[D_1(2430)] = (384^{+130}_{-110})$ MeV
  - ☞ $\boldsymbol{s}_\ell = \boldsymbol{s}_q + \boldsymbol{L}$ $\Rightarrow$ for P-wave charmed mesons: $s_\ell^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$
  - ☞ for decays $D_1 \to D^*\pi$:

    $\frac{1}{2}^+ \to \frac{1}{2}^- + 0^-$ in S-wave $\Rightarrow$ large width

    $\frac{3}{2}^+ \to \frac{1}{2}^- + 0^-$ in D-wave $\Rightarrow$ small width
  - ☞ thus, $D_1(2420)$: $s_\ell^P = \frac{3}{2}^+$, $D_1(2430)$: $s_\ell^P = \frac{1}{2}^+$
- Suppression of the S-wave production of $\frac{3}{2}^+ + \frac{1}{2}^-$ heavy meson pairs in e^+e^- annihilation Table VI in E.Eichten et al., PRD**17**(1978)3090; X. Li, M. Voloshin, PRD**88**(2013)034012

Exercise: Try to understand this statement as a consequence of HQSS.

Applications of HQS: $D^*_{s0}(2317)$ and $D_{s1}(2460)$ (1)

- HQFS: for a singly-heavy hadron,

$$M_{H_Q} = m_Q + A + \mathcal{O}\left(\frac{\Lambda^2_{\text{QCD}}}{m_Q}\right) \text{ with } A \text{ independent of } m_Q$$

- rough estimates of bottom analogues whatever the D_{sJ} states are

$$M_{B^*_{s0}} = M_{D^*_{s0}(2317)} + \Delta_{b-c} + \mathcal{O}\left(\Lambda^2_{\text{QCD}}\left(\frac{1}{m_c} - \frac{1}{m_b}\right)\right) \simeq (5.65 \pm 0.15)\ \text{GeV}$$

$$M_{B_{s1}} = M_{D_{s1}(2460)} + \Delta_{b-c} + \mathcal{O}\left(\Lambda^2_{\text{QCD}}\left(\frac{1}{m_c} - \frac{1}{m_b}\right)\right) \simeq (5.79 \pm 0.15)\ \text{GeV}$$

here $\Delta_{b-c} \equiv m_b - m_c \simeq \overline{M}_{B_s} - \overline{M}_{D_s} \simeq 3.33$ GeV, where $\overline{M}_{B_s} = 5.403$ GeV, $\overline{M}_{D_s} = 2.076$ GeV: spin-averaged g.s. $Q\bar{s}$ meson masses

☞ both to be discovered [1]

- Lattice QCD results: Lang, Mohler, Prelovsek, Woloshyn, PLB750(2015)17

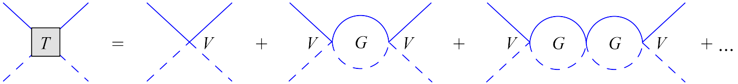
$$M^{\text{lat.}}_{B^*_{s0}} = (5.711 \pm 0.013 \pm 0.019)\ \text{GeV}$$
$$M^{\text{lat.}}_{B_{s1}} = (5.750 \pm 0.017 \pm 0.019)\ \text{GeV}$$

[1] The established meson $B_{s1}(5830)$ is probably the bottom partner of $D_{s1}(2536)$.

Applications of HQS: $D^*_{s0}(2317)$ and $D_{s1}(2460)$ (2)

- in hadronic molecular model: $D^*_{s0}(2317)[\simeq DK]$, $D_{s1}(2460)[\simeq D^*K]$

Barnes, Close, Lipkin (2003); van Beveren, Rupp (2003); Kolomeitsev, Lutz (2004); FKG et al. (2006); ...



$D^{(*)}K$ bound states: poles of the T-matrix

- HQSS $\Rightarrow$ similar binding energies $M_D + M_K - M_{D^*_{s0}} \simeq 45$ MeV

$$M_{D_{s1}(2460)} - M_{D^*_{s0}(2317)} \simeq M_{D^*} - M_D \text{ is natural}$$

- HQFS $\Rightarrow$ predicting the 0^+ and 1^+ bottom-partner masses

$$M_{B^*_{s0}} \simeq M_B + M_K - 45\,\text{MeV} \simeq 5.730\text{ GeV}$$
$$M_{B_{s1}} \simeq M_{B^*} + M_K - 45\,\text{MeV} \simeq 5.776\text{ GeV}$$

FKG, Meißner, Zou, *How the X(5568) challenges our understanding of QCD*, Commun.Theor.Phys. 65 (2016) 593

- mass too low for $X(5568)$ to be a $\bar{b}s\bar{u}d$: $M \simeq M_{B_s} + 200$ MeV
  - ☞ $M_\pi \simeq 140$ MeV because pions are pseudo-Goldstone bosons
  - ☞ Gell-Mann–Oakes–Renner: $M_\pi^2 \propto m_q$
  - ☞ For any matter field: $M_R \gg M_\pi$; one expects $M_{\bar{u}d} \sim M_R \gtrsim M_\sigma$

$$M_{\bar{b}s\bar{u}d} \gtrsim M_{B_s} + 500 \text{ MeV} \sim 5.9 \text{ GeV}$$

- HQFS predicts an isovector X_c:

$$M_{X_c} = M_{X(5568)} - \Delta_{b-c} + \mathcal{O}\left(\Lambda_{\text{QCD}}^2\left(\frac{1}{m_c} - \frac{1}{m_b}\right)\right) \simeq (2.24 \pm 0.15) \text{ GeV}$$

but in $D_s\pi$, only the isoscalar $D_{s0}^*(2317)$ was observed! BaBar (2003)

☞ properties of $X(5568)$ hard to understand

- negative results reported by LHCb, LHCb, PRL117(2016)152003
  by CMS, CMS, PRL120(2018)202005
  by CDF, CDF, PRL120(2018)202006
  by ATLAS ATLAS, PRL120(2018)202007

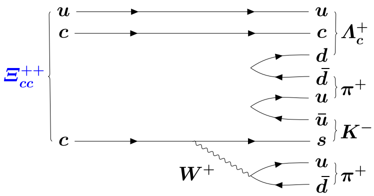
Development inspired by the LHCb discovery of the $\Xi_{cc}(3620)^{++}$

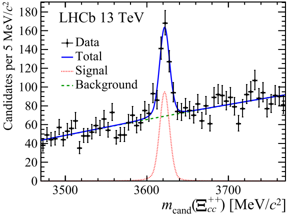
- Heavy antiquark-diquark symmetry (HADS):
  repalcing $\bar{Q}$ in $\bar{Q}q$ by $QQ \quad \Rightarrow \quad QQq$;
  repalcing $\bar{Q}$ in $\bar{Q}\bar{q}\bar{q}$ by $QQ \quad \Rightarrow \quad QQ\bar{q}\bar{q}$;

$$\begin{array}{cccccccc} & \bar{Q}q & \Rightarrow & QQq, & \bar{Q}\bar{q}\bar{q} & \Rightarrow & QQ\bar{q}\bar{q} \\ M: & m_Q + A & \Rightarrow & m_{QQ} + A, & m_Q + B & \Rightarrow & m_{QQ} + B \end{array}$$

  Prediction: $\quad M_{QQ\bar{q}\bar{q}} - M_{\bar{Q}\bar{q}\bar{q}} \simeq M_{QQq} - M_{\bar{Q}q}$

- Doubly-charmed baryon discovered by LHCb PRL119(2017)112001 [arXiv:1707.01621]





$M_{\Xi_{cc}^{++}} = (3621.40 \pm 0.78)$ MeV can be used as input

From heavy baryons to doubly-heavy tetraquarks (2)

TABLE II. Expectations for the ground-state tetraquark masses, in MeV.[a] The column labeled "HQS relation" is the result of our heavy-quark symmetry relations and is explicitly given by the sum of the right-hand side of Eq. (1) and the kinetic-energy mass shifts of Eq. (7). Here q denotes an up or down quark. For stable tetraquark states the $\mathcal{Q}$ value is highlighted in a box.

State	J^P	j_ℓ	$m(Q_iQ_jq_m)$ (c.g.)	HQS relation	$m(Q_iQ_j\bar{q}_k\bar{q}_l)$	Decay channel	$\mathcal{Q}$ (MeV)
$\{cc\}[\bar{u}\,\bar{d}]$	1^+	0	3663[b]	$m(\{cc\}u)+315$	3978	D^+D^{*0} 3876	102
$\{cc\}[\bar{q}_k\bar{s}]$	1^+	0	3764[c]	$m(\{cc\}s)+392$	4156	$D^+D_s^{*-}$ 3977	179
$\{cc\}\{\bar{q}_k\bar{q}_l\}$	$0^+,1^+,2^+$	1	3663	$m(\{cc\}u)+526$	4146,4167,4210	D^+D^0, D^+D^{*0} 3734,3876	412,292,476
$[bc][\bar{u}\,\bar{d}]$	0^+	0	6914	$m([bc]u)+315$	7229	B^-D^+/B^0D^0 7146	83
$[bc][\bar{q}_k\bar{s}]$	0^+	0	7010[d]	$m([bc]s)+392$	7406	B_sD 7236	170
$[bc]\{\bar{q}_k\bar{q}_l\}$	1^+	1	6914	$m([bc]u)+526$	7439	B^*D/BD^* 7190/7290	249
$\{bc\}[\bar{u}\,\bar{d}]$	1^+	0	6957	$m(\{bc\}u)+315$	7272	B^*D/BD^* 7190/7290	82
$\{bc\}[\bar{q}_k\bar{s}]$	1^+	0	7053[d]	$m(\{bc\}s)+392$	7445	DB_s^* 7282	163
$\{bc\}\{\bar{q}_k\bar{q}_l\}$	$0^+,1^+,2^+$	1	6957	$m(\{bc\}u)+526$	7461,7472,7493	BD/B^*D 7146/7190	317,282,349
$\{bb\}[\bar{u}\,\bar{d}]$	1^+	0	10 176	$m(\{bb\}u)+306$	10 482	$B^-\bar{B}^{*0}$ 10 603	$\boxed{-121}$
$\{bb\}[\bar{q}_k\bar{s}]$	1^+	0	10 252[c]	$m(\{bb\}s)+391$	10 643	$\bar{B}\bar{B}_s^*/\bar{B}_s\bar{B}^*$ 10 695/10 691	$\boxed{-48}$
$\{bb\}\{\bar{q}_k\bar{q}_l\}$	$0^+,1^+,2^+$	1	10 176	$m(\{bb\}u)+512$	10 674,10 681,10 695	B^-B^0, B^-B^{*0} 10 559,10 603	115,78,136

[a]Masses of the unobserved doubly heavy baryons are taken from Ref. [14]; for lattice evaluations of b-baryon masses, see Ref. [15].
[b]Based on the mass of the LHCb Ξ_{cc}^{++} candidate, 3621.40 MeV, Ref. [10].
[c]Using the s/d mass differences of the corresponding heavy-light mesons.
[d]Evaluated as $\frac{1}{2}[m(c\bar{s})-m(c\bar{d})+m(b\bar{s})-m(b\bar{d})]+m(bcd)$.

Eichten, Quigg, PRL119(2017)202002

- HADS ⇒ stable doubly-bottom tetraquarks (only decay weakly) are likely to exist

  see also Carlson, Heller, Tjon, PRD37(1988)744; Manohar, Wise, NPB399(1993)17; Karliner, Rosner, PRL119(2017)202001; Czarnecki, Leng, Voloshin, PLB778(2018)233; ...

- support from lattice QCD — Francis, Hudspith, Lewis, Maltman, PRL118(2017)142001

HQS for XYZ states

HQSS for XYZ (1)

- Assuming the $X(3872)$ to be a $D\bar{D}^*$ molecule
- Consider S-wave interaction between a pair of $s_\ell^P = \frac{1}{2}^-$ (anti-)heavy mesons:

$$\begin{aligned} 0^{++}: &\quad D\bar{D}, \quad D^*\bar{D}^* \\ 1^{+-}: &\quad \frac{1}{\sqrt{2}}\left(D\bar{D}^* + D^*\bar{D}\right), \quad D^*\bar{D}^* \\ 1^{++}: &\quad \frac{1}{\sqrt{2}}\left(D\bar{D}^* - D^*\bar{D}\right) \\ 2^{++}: &\quad D^*\bar{D}^* \end{aligned}$$

here, charge conjugation phase convention: $D \overset{C}{\to} +\bar{D},\ D^* \overset{C}{\to} -\bar{D}^*$

- Heavy quark spin irrelevant $\Rightarrow$ interaction matrix elements:

$$\left\langle s_{1\,\ell}, s_{2\,\ell}, s_L \left| \hat{\mathcal{H}} \right| s'_{1\,\ell}, s'_{2\,\ell}, s_L \right\rangle$$

For each isospin, 2 independent terms

$$\left\langle \frac{1}{2}, \frac{1}{2}, 0 \left| \hat{\mathcal{H}} \right| \frac{1}{2}, \frac{1}{2}, 0 \right\rangle, \qquad \left\langle \frac{1}{2}, \frac{1}{2}, 1 \left| \hat{\mathcal{H}} \right| \frac{1}{2}, \frac{1}{2}, 1 \right\rangle$$

$\Rightarrow$ 6 pairs grouped in 2 multiplets with $s_L = 0$ and 1, respectively

HQSS for XYZ (2)

- For the HQSS consequences, convenient to use the basis of states: $s_L^{PC} \otimes s_{c\bar{c}}^{PC}$
  - ☞ S-wave: $s_L^{PC}, s_{c\bar{c}}^{PC} = 0^{-+}$ or 1^{--}
  - ☞ multiplet with $s_L = 0$:

$$0_L^{-+} \otimes 0_{c\bar{c}}^{-+} = 0^{++}, \qquad 0_L^{-+} \otimes 1_{c\bar{c}}^{--} = 1^{+-}$$

  - ☞ multiplet with $s_L = 1$:

$$1_L^{--} \otimes 0_{c\bar{c}}^{-+} = 1^{+-}, \qquad 1_L^{--} \otimes 1_{c\bar{c}}^{--} = 0^{++} \oplus \boxed{\mathbf{1^{++}}} \oplus 2^{++}$$

- Multiplets in strict heavy quark limit:
  - ☞ $X(3872)$ has three partners with 0^{++}, 2^{++} and 1^{+-}

Hidalgo-Duque et al., PLB727(2013)432; Baru et al., PLB763(2016)20

  - ☞ might be 6 $I = 1$ molecules:

$Z_b[1^{+-}], Z_b'[1^{+-}]$ and $W_{b0}[0^{++}]$, $W_{b0}'[0^{++}]$, $W_{b1}[1^{++}]$ and $W_{b2}[2^{++}]$

Bondar et al., PRD84(2011)054010; Voloshin, PRD84(2011)031502;
Mehen, Powell, PRD84(2011)114013

- Recall the excercise in Lecture-1:

  Is $\Upsilon\pi^+\pi^-$ a good choice of final states for the search of X_b, the $J^{PC}=1^{++}$ bottom analogue of the $X(3872)$?

  Answer: No. $X_b \to \Upsilon\pi\pi$ breaks isospin symmetry

  FKG, Hidalgo-Duque, Nieves, Valderrama, PRD88(2013)054007; Karliner, Rosner, PRD91(2015)014014

  $M_{B^0} - M_{B^\pm} = (0.31 \pm 0.06)$ MeV   [$M_{D^\pm} - M_{D^0} = (4.822 \pm 0.015)$ MeV]

- Negative results:

  CMS, *Search for a new bottomonium state decaying to $\Upsilon(1S)\pi^+\pi^-$ in pp collisions at $\sqrt{s}=8$ TeV*, PLB727(2013)57;

  ATLAS, *Search for the X_b and other hidden-beauty states in the $\pi^+\pi^-\Upsilon(1S)$ channel at ATLAS*, PLB740(2015)199

- The results can be reinterpreted as for the search of W_{bJ} $(I=1, J^{++})$

$$1_L^{--} \otimes 1_{c\bar c}^{--} = 0^{++} \oplus \boxed{\mathbf{1}^{++}} \oplus 2^{++}$$

- Heavy quark spin selection rule for $X(3872)$: Voloshin, PLB604(2004)69
  for $X(3872)$ being a $\mathbf{1}^{++}$ $D\bar D^*$ molecule, $s_L = 1$, $s_{c\bar c} = 1$
- spin structure of $Q\bar Q$:

	s_L	$s_{c\bar c}$	J^{PC}	$c\bar c$
S-wave	0	0	0^{-+}	η_c
	0	1	1^{--}	J/ψ
P-wave	1	0	1^{+-}	h_c
	1	1	$(0,1,2)^{++}$	$\chi_{c0}, \chi_{c1}, \chi_{c2}$

- allowed: $X(3872) \to J/\psi\pi\pi$, $X(3872) \to \chi_{cJ}\pi$, $X(3872) \to \chi_{cJ}\pi\pi$
  suppressed: $X(3872) \to \eta_c\pi\pi$, $X(3872) \to h_c\pi\pi$
- Interesting feature of $Z_b^{(\prime)}$: observed with similar rates in both $\Upsilon\pi\pi[s_{b\bar b} = 1]$ and $h_b\pi\pi[s_{b\bar b} = 0]$ Bondar, Garmash, Milstein, Mizuk, Voloshin, PRD84(2011)054010

  $Z_b \sim B\bar B^* \sim 0_{b\bar b}^- \otimes 1_{q\bar q}^- - 1_{b\bar b}^- \otimes 0_{q\bar q}^-$, $\quad Z_b' \sim B^*\bar B^* \sim 0_{b\bar b}^- \otimes 1_{q\bar q}^- + 1_{b\bar b}^- \otimes 0_{q\bar q}^-$

Unitary transformation from two-meson basis to $\left|s_{1\,c}, s_{2\,c}, s_{c\bar{c}}; s_{1\,\ell}, s_{2\,\ell}, s_L; J\right\rangle$:

$$\left|s_{1\,c}, s_{1\,\ell}, j_1; s_{2\,c}, s_{2\,\ell}, j_2; J\right\rangle = \sum_{s_{c\bar{c}}, s_L} \sqrt{(2j_1+1)(2j_2+1)(2s_{c\bar{c}}+1)(2s_L+1)} \times \begin{Bmatrix} s_{1\,c} & s_{2\,c} & s_{c\bar{c}} \\ s_{1\,\ell} & s_{2\,\ell} & s_L \\ j_1 & j_2 & J \end{Bmatrix} \left|s_{1\,c}, s_{2\,c}, s_{c\bar{c}}; s_{1\,\ell}, s_{2\,\ell}, s_L; J\right\rangle$$

$j_{1,2}$: meson spins;

J: the total angular momentum of the whole system

$s_{1\,c(2\,c)} = \frac{1}{2}$: spin of the heavy quark in meson 1 (2)

$s_{1\,\ell(2\,\ell)} = \frac{1}{2}$: angular momentum of the light quarks in meson 1 (2)

- $s_{c\bar{c}} = 0, 1$: total spin of $c\bar{c}$, conserved but decoupled
- $s_L = 0, 1$: total angular momentum of the light-quark system, conserved
- only two independent $\left\langle s_{\ell 1}, s_{\ell 2}, s_L \middle| \hat{\mathcal{H}} \middle| s'_{\ell 1}, s'_{\ell 2}, s_L \right\rangle_I$ terms for each isospin I:

$$F_{I0} = \left\langle \tfrac{1}{2}, \tfrac{1}{2}, 0 \middle| \hat{\mathcal{H}} \middle| \tfrac{1}{2}, \tfrac{1}{2}, 0 \right\rangle_I, \quad F_{I1} = \left\langle \tfrac{1}{2}, \tfrac{1}{2}, 1 \middle| \hat{\mathcal{H}} \middle| \tfrac{1}{2}, \tfrac{1}{2}, 1 \right\rangle_I$$

$$\begin{pmatrix} D\bar{D} \\ D^*\bar{D}^* \end{pmatrix}: \qquad V^{(0^{++})} = \begin{pmatrix} C_{IA} & \sqrt{3}C_{IB} \\ \sqrt{3}C_{IB} & C_{IA} - 2C_{IB} \end{pmatrix},$$

$$\begin{pmatrix} D\bar{D}^* \\ D^*\bar{D}^* \end{pmatrix}: \qquad V^{(1^{+-})} = \begin{pmatrix} C_{IA} - C_{IB} & 2C_{IB} \\ 2C_{IB} & C_{IA} - C_{IB} \end{pmatrix},$$

$$D\bar{D}^*: \qquad V^{(1^{++})} = C_{IA} + C_{IB},$$

$$D^*\bar{D}^*: \qquad V^{(2^{++})} = C_{IA} + C_{IB},$$

here, $C_{IA} = \frac{1}{4}(3F_{I1} + F_{I0})$, $C_{IB} = \frac{1}{4}(F_{I1} - F_{I0})$

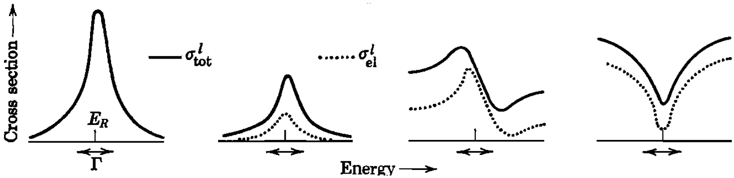
- This predicts a spin partner for $X(3872)$: Nieves, Valderrama, PRD86(2012)056004; ...

$$M_{X_2(4013)} - M_{X(3872)} \approx M_{D^*} - M_D$$

ongoing efforts searching for X_2, not found yet

Threshold cusps and triangle singularties

Resonances do not always appear as peaks:



J. R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions*

Peaks are not always due to resonances:

- Dynamics ⇒ poles in the S-matrix (resonances): genuine physical states.
- Kinematic effects ⇒ branching points of S-matrix
  - ☞ normal two-body threshold cusp
  - ☞ triangle singularity
  - ☞ ...

tools/traps in hadron spectroscopy

- Unitarity of the S-matrix: $S\,S^\dagger = S^\dagger S = \mathbb{1}$, $S_{fi} = \delta_{fi} - i(2\pi)^4\delta^4(p_f - p_i)T_{fi}$

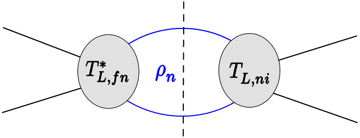
  T-matrix: $$T_{fi} - T^\dagger_{fi} = -i(2\pi)^4 \underbrace{\sum_n}_{\text{all physically accesible states}} \delta(p_n - p_i)T^\dagger_{fn}T_{ni}$$

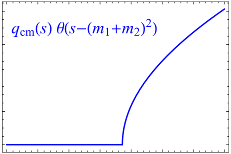
  assuming all intermediate states are two-body, partial-wave unitarity relation:

  $$\boxed{\text{Im}\, T_{L,fi}(s) = -\sum_n T^*_{L,fn}\,\rho_n(s)\,T_{L,ni}}$$

  2-body phase space factor: $\rho_n(s) = q_{\text{cm},n}(s)/(2\sqrt{s})\theta(\sqrt{s} - m_{n1} - m_{n2})$,

  $q_{\text{cm},n}(s) = \sqrt{[s-(m_{n1}+m_{n2})^2][s-(m_{n1}-m_{n2})^2]}/(2\sqrt{s})$

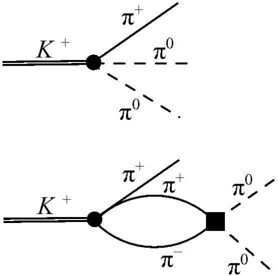


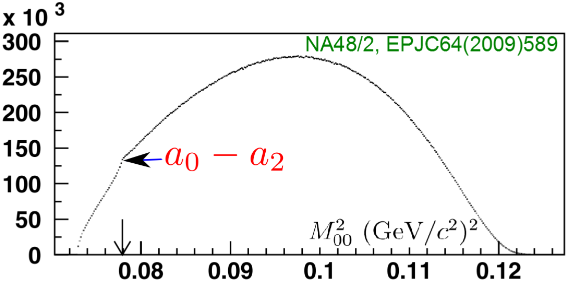


- There is always a cusp at an S-wave threshold

- Cusp effect as a useful tool for precise measurement:
  - ☞ example of the cusp in $K^\pm \to \pi^\pm\pi^0\pi^0$
  - ☞ strength of the cusp measures the interaction strength!

Meißner, Müller, Steininger (1997); Cabibbo (2004); Colangelo, Gasser, Kubis, Rusetsky (2006); . . .

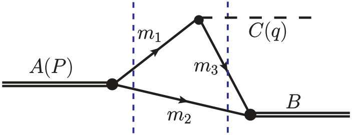




$\sim$ threshold, only sensitive to scattering length, $(a_0 - a_2)M_{\pi^+} = 0.2571 \pm 0.0056$

- Very prominent cusp $\Rightarrow$ large scattering length $\Rightarrow$ likely a nearby pole

  effective range expansion (ERE): $f(k) = \frac{1}{1/a + rk^2/2 - ik}$



$$\frac{1}{2m_A}\sqrt{\lambda(m_A^2, m_1^2, m_2^2)} \equiv \boxed{p_{2,\text{left}} = p_{2,\text{right}}} \equiv \gamma\,(\beta\, E_2^* - p_2^*)$$

on-shell momentum of m_2 at the left and right cuts in the A rest frame

$\beta = |\vec{p}_{23}|/E_{23}, \gamma = 1/\sqrt{1-\beta^2}$ Bayar et al., PRD94(2016)074039

- $p_2 > 0,\, p_3 = \gamma\,(\beta\, E_3^* + p_2^*) > 0 \Rightarrow m_2$ and m_3 move in the same direction
- velocities in the A rest frame: $v_3 > \beta > v_2$

$$v_2 = \beta\,\frac{E_2^* - p_2^*/\beta}{E_2^* - \beta\, p_2^*} < \beta\,, \qquad v_3 = \beta\,\frac{E_3^* + p_2^*/\beta}{E_3^* + \beta\, p_2^*} > \beta$$

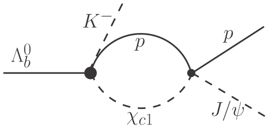
- Conditions (Coleman–Norton theorem): Coleman, Norton (1965); Bronzan (1964)
  ☞ all three intermediate particles can go on shell simultaneously
  ☞ $\vec{p}_2 \parallel \vec{p}_3$, particle-3 can catch up with particle-2 (as a classical process)
- needs very special kinematics $\Rightarrow$ process dependent! (contrary to pole position)

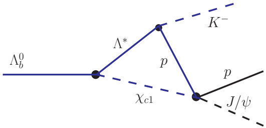
Coincidence of $P_c(4450)$ with kinematic singularities

- Mass: $M_{P_c(4450)} = (4449.8 \pm 1.7 \pm 2.5)$ MeV
- Our trivial observation: $P_c(4450)$ coincides with the $\chi_{c1}p$ threshold:

$$M_{P_c(4450)} - M_{\chi_{c1}} - M_p = (0.9 \pm 3.1)\ \text{MeV}$$

- Our non-trivial observation: there is a triangle singularity at the same time!
  Solving the equation $\boxed{p_{2,\text{left}} = p_{2,\text{right}}}$
  $\Rightarrow$
  to have a TS at $M_{J/\psi p} = M_{\chi_{c1}} + M_p$, we need $M_{\Lambda^*} \simeq 1.89$ GeV
  On shell $\Rightarrow \Lambda^*$ must be unstable, the TS is then a finite peak





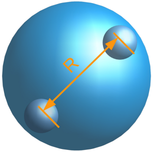
More possible relevant TSs, see X.-H. Liu, Q. Wang, Q. Zhao, PLB757(2015)231

Compositeness and hadronic molecules

FKG, Hanhart, Meißner, Wang, Zhao, Zou, *Hadronic molecules*, Rev. Mod. Phys. **90** (2018) 015004

- Hadronic molecule:
  dominant component is a composite state of 2 or more hadrons
- Concept at large distances, so that can be approximated by system of multi-hadrons at low energies
  Consider a 2-body bound state with a mass $M = m_1 + m_2 - E_B$

size: $R \sim \frac{1}{\sqrt{2\mu E_B}} \gg r_{\text{hadron}}$



- scale separation $\Rightarrow$ power expansion in p/Λ, (nonrelativistic) EFT applicable!
- Only narrow hadrons can be considered as components of hadronic molecules, $\Gamma_h \ll 1/r$, r: range of forces

Filin *et al.*, PRL105(2010)019101; FKG, Meißner, PRD84(2011)014013

- Why are hadronic molecules interesting?
  - ☞ one realization of color-neutral objects, analogue of light nuclei
  - ☞ important information for hadron-hadron interaction
  - ☞ understanding the XYZ states
  - ☞ EFT applicable; model-independent statements can be made for S-wave, compositeness $(1-Z)$ related to measurable quantities compositeness: probability of the physical state being a 2-body bound state

Weinberg, PR137(1965); Baru *et al.*, PLB586(2004); Hyodo, IJMPA28(2013)1330045; ...

see also, e.g., Weinberg's books: QFT Vol.I, Lectures on QM

$$|g_{\rm NR}|^2 \approx (1-Z)\frac{2\pi}{\mu^2}\sqrt{2\mu E_B} \leq \frac{2\pi}{\mu^2}\sqrt{2\mu E_B}$$

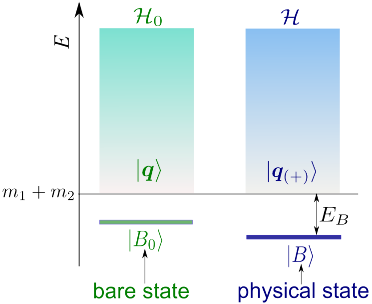
$$a \approx -\frac{2(1-Z)}{(2-Z)\sqrt{2\mu E_B}}, \quad r_e \approx \frac{Z}{(1-Z)\sqrt{2\mu E_B}}$$

Model-independent result for S-wave loosely bound composite states:

Consider a system with Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + V$$

$\mathcal{H}_0$: free Hamiltonian, V: interaction potential

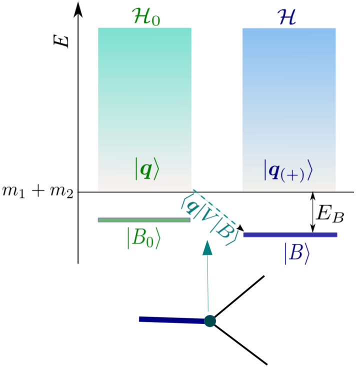


- Compositeness: the probability of finding the physical state $|B\rangle$ in the 2-body continuum $|\boldsymbol{q}\rangle$

$$1 - Z = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \left|\langle \boldsymbol{q}|B\rangle\right|^2$$

- $Z = \left|\langle B_0|B\rangle\right|^2, \qquad 0 \le (1-Z) \le 1$
  - ☞ $Z = 0$: pure bound (composite) state
  - ☞ $Z = 1$: pure elementary state

Compositeness : $1 - Z = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} |\langle \boldsymbol{q}|B\rangle|^2$



- Schrödinger equation

$$(\mathcal{H}_0 + V)|B\rangle = -E_B|B\rangle$$

multiplying by $\langle \boldsymbol{q}|$ and using $\mathcal{H}_0|\boldsymbol{q}\rangle = \frac{\boldsymbol{q}^2}{2\mu}|\boldsymbol{q}\rangle$:
$\Rightarrow$ momentum-space wave function:

$$\langle \boldsymbol{q}|B\rangle = -\frac{\langle \boldsymbol{q}|V|B\rangle}{E_B + \boldsymbol{q}^2/(2\mu)}$$

- S-wave, small binding energy so that $R = 1/\sqrt{2\mu E_B} \gg r$, r: range of forces

$$\langle \boldsymbol{q}|V|B\rangle = g_{\rm NR}\left[1 + \mathcal{O}(r/R)\right]$$

- Compositeness:

$$1 - Z = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{|g_{\rm NR}|^2}{\left[E_B + \boldsymbol{q}^2/(2\mu)\right]^2}\left[1 + \mathcal{O}\left(\frac{r}{R}\right)\right] = \frac{\mu^2|g_{\rm NR}|^2}{2\pi\sqrt{2\mu E_B}}\left[1 + \mathcal{O}\left(\frac{r}{R}\right)\right]$$

- Coupling constant measures the compositeness for an S-wave shallow bound state

$$|g_{\rm NR}|^2 \approx (1-Z)\frac{2\pi}{\mu^2}\sqrt{2\mu E_B} \leq \frac{2\pi}{\mu^2}\sqrt{2\mu E_B}$$

bounded from the above

Exercise:

Show that $|g_{\rm NR}|^2$ is the residue of the T-matrix element at the pole $E=-E_B$:

$$|g_{\rm NR}|^2 = \lim_{E\to -E_B}(E+E_B)\langle \boldsymbol{k}|T|\boldsymbol{k}\rangle$$

Hint: use the Lippmann–Schwinger equation $T = V + V\frac{1}{E-\mathcal{H}_0+i\epsilon}T$ and the completeness relation $|B\rangle\langle B| + \int\frac{d^3q}{(2\pi)^3}|\boldsymbol{q}_{(+)}\rangle\langle \boldsymbol{q}_{(+)}| = 1$ to derive the Low equation (noticing $T|\boldsymbol{q}\rangle = V|\boldsymbol{q}_{(+)}\rangle$):

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle + \frac{\langle \boldsymbol{k}'|V|B\rangle\langle B|V|\boldsymbol{k}\rangle}{E+E_B+i\epsilon} + \int\frac{d^3q}{(2\pi)^3}\frac{\langle \boldsymbol{k}'|T|\boldsymbol{q}\,\rangle\langle \boldsymbol{q}\,|T^\dagger|\boldsymbol{k}\rangle}{E-\boldsymbol{q}^2/(2\mu)+i\epsilon}$$

Compositeness (4)

- Z can be related to scattering length a and effective range r_e Weinberg (1965)

$$a = -\frac{2R(1-Z)}{2-Z}\left[1+\mathcal{O}\left(\frac{r}{R}\right)\right], \quad r_e = \frac{RZ}{1-Z}\left[1+\mathcal{O}\left(\frac{r}{R}\right)\right]$$

Effective range expansion: $f^{-1}(k) = 1/a + r_e k^2/2 - ik + \mathcal{O}\left(k^4\right)$

<u>Derivation</u>:

$$T(E) \equiv \langle k|T|k\rangle = -\frac{2\pi}{\mu} f(k) \qquad \Rightarrow \qquad \mathrm{Im}\, T^{-1}(E) = \frac{\mu}{2\pi}\sqrt{2\mu E}\,\theta(E)$$

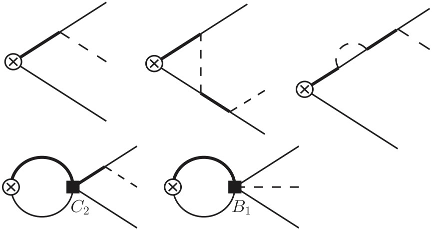
Twice-subtracted dispersion relation for $T^{-1}(E)$

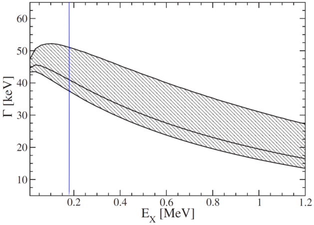
$$\begin{aligned} T^{-1}(E) &= \frac{E+E_B}{|g_{\mathrm{NR}}|^2} + \frac{(E+E_B)^2}{\pi}\int_0^{+\infty} dw \frac{\mathrm{Im}\, T^{-1}(w)}{(w-E-i\epsilon)(w+E_B)^2} \\ &= \frac{E+E_B}{|g_{\mathrm{NR}}|^2} + \frac{\mu\, R}{4\pi}\left(\frac{1}{R} - \sqrt{-2\mu E - i\epsilon}\right)^2 \end{aligned}$$

- <u>Example</u>: deuteron as pn bound state. Exp.: $E_B = 2.2$ MeV, $a_{^3S_1} = -5.4$ fm

$$a_{Z=1} = 0 \text{ fm}, \qquad a_{Z=0} = (-4.3 \pm 1.4) \text{ fm}$$

- Coupling constant fixed by binding energy, long-distance processes such as $X(3872) \to D^0\bar{D}^0\pi^0, D^0\bar{D}^0\gamma$ calculable
  E.g., XEFT prediction of $\Gamma(X \to D^0\bar{D}^0\pi^0)$ Fleming et al., PRD76(2007)034006





- compositeness from scattering length:
  scattering lengths calculable using the Lüscher formalism in lattice QCD
  E.g., from DK $I=0$ scattering length $\Rightarrow$ $D^*_{s0}(2317)$ contains $\gtrsim 70\%DK$

Liu et al., PRD86(2013)014508; Martínez Torres et al., JHEP1505,053; Bali et al., PRD96(2017)074501

Summary

- Lots of resonances or resonance-like structures observed in recent years, many puzzles
- QCD symmetries (chiral, heavy quark) prove to be useful tools
- Many more data needed, lots of work needs to be done

Thank you for your attention!

Backup slides

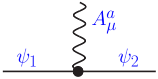
Symmetry implies a derivative coupling for GBs, i.e.,

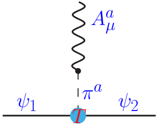
$$\boxed{\text{GBs do not interact at vanishing momenta}}$$

- Consider GB π^a: $\langle \pi^a | Q_A^a | 0 \rangle = \int d^3x \, \langle \pi^a | A_0^a(x) | 0 \rangle \neq 0$

  Lorentz invariance $\Rightarrow$ $\big\langle \pi^a(q) | A_\mu^a(0) | 0 \big\rangle = -i q_\mu F_\pi$

- Consider the matrix element

$\langle \psi_1 | A_\mu^a(0) | \psi_2 \rangle =$ 

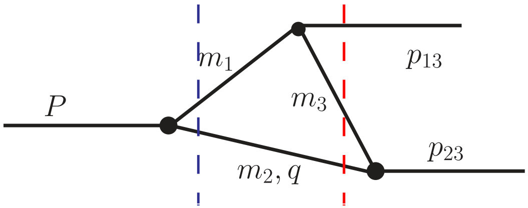
$+$ 

$$= \quad R_\mu^a \quad + \quad F_\pi q^\mu \frac{1}{q^2} T^a$$

  Current conservation $\Rightarrow$ $q^\mu A_\mu^a = 0$, thus

$$q^\mu R_\mu^a + F_\pi T^a = 0 \quad \Rightarrow \quad \lim_{q^\mu \to 0} T^a = 0$$

- $\Rightarrow$ GBs couple in a derivative form !!



Consider the scalar three-point loop integral

$$I = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{[(P-q)^2 - m_1^2 + i\epsilon]\,(q^2 - m_2^2 + i\epsilon)\,[(p_{23}-q)^2 - m_3^2 + i\epsilon]}$$

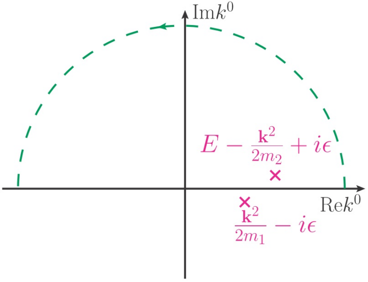
Rewriting a propagator into two poles:

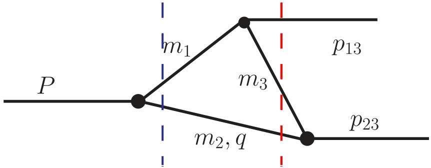
$$\frac{1}{q^2 - m_2^2 + i\epsilon} = \frac{1}{(q^0 - \omega_2 + i\epsilon)\,(q^0 + \omega_2 - i\epsilon)} \quad \text{with} \quad \omega_2 = \sqrt{m_2^2 + \vec{q}^{\,2}}$$

focus on the positive-energy poles

$$I \simeq \frac{i}{8m_1m_2m_3} \int \frac{dq^0 d^3\vec{q}}{(2\pi)^4} \frac{1}{(P^0 - q^0 - \omega_1 + i\epsilon)\,(q^0 - \omega_2 + i\epsilon)\,(p_{23}^0 - q^0 - \omega_3 + i\epsilon)}$$

TS: some details (2)





Contour integral over $q^0 \Rightarrow$ cut-1 cut-2

$$I \propto \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{[P^0 - \omega_1(q) - \omega_2(q) + i\,\epsilon][p_{23}^0 - \omega_2(q) - \omega_3(\vec{p}_{23} - \vec{q}) + i\,\epsilon]}$$

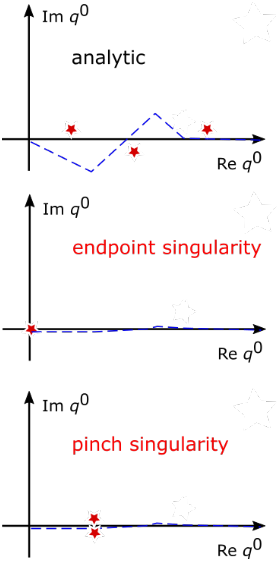
$$\propto \int_0^\infty dq\, \frac{q^2}{P^0 - \omega_1(q) - \omega_2(q) + i\,\epsilon} f(q)$$

The second cut:

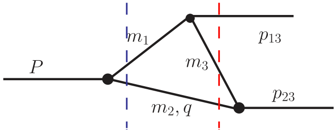
$$f(q) = \int_{-1}^{1} dz\, \frac{1}{p_{23}^0 - \omega_2(q) - \sqrt{m_3^2 + q^2 + p_{23}^2 - 2p_{23}qz} + i\,\epsilon}$$

Relation between singularities of integrand and integral

- singularity of integrand does not necessarily give a singularity of integral:
  integral contour may be deformed to avoid the singularity
- Two cases that a singularity cannot be avoided:
  - ☞ endpoint singularity
  - ☞ pinch singularity



TS: some details (4)



$$I \propto \int_0^\infty dq \, \frac{q^2}{P^0 - \omega_1(q) - \omega_2(q) + i\,\epsilon} f(q)$$

$$f(q) = \int_{-1}^{1} dz \, \frac{1}{A(q,z)} \equiv \int_{-1}^{1} dz \, \frac{1}{p_{23}^0 - \omega_2(q) - \sqrt{m_3^2 + q^2 + p_{23}^2 - 2p_{23}qz} + i\,\epsilon}$$

Singularities of the **integrand of I** in the rest frame of initial particle ($P^0 = M$):

- 1st cut: $M - \omega_1(l) - \omega_2(l) + i\,\epsilon = 0 \Rightarrow$
$$q_{\text{on}\pm} \equiv \pm \left(\frac{1}{2M} \sqrt{\lambda(M^2, m_1^2, m_2^2)} + i\,\epsilon \right)$$
- 2nd cut: $A(q, \pm 1) = 0 \Rightarrow$ endpoint singularities of $f(q)$

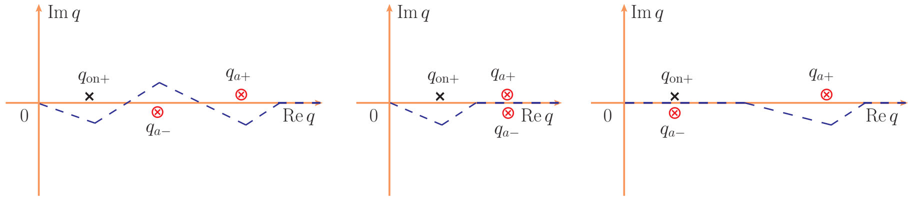
$$z = +1: \quad q_{a+} = \gamma\,(\beta\,E_2^* + p_2^*) + i\,\epsilon\,, \qquad q_{a-} = \gamma\,(\beta\,E_2^* - p_2^*) - i\,\epsilon\,,$$
$$z = -1: \quad q_{b+} = \gamma\,(-\beta\,E_2^* + p_2^*) + i\,\epsilon\,, \quad q_{b-} = -\gamma\,(\beta\,E_2^* + p_2^*) - i\,\epsilon$$
$$\beta = |\vec{p}_{23}|/E_{23}, \qquad \gamma = 1/\sqrt{1-\beta^2} = E_{23}/m_{23}$$

$E_2^*(p_2^*)$: energy (momentum) of particle-2 in the cmf of the (2,3) system

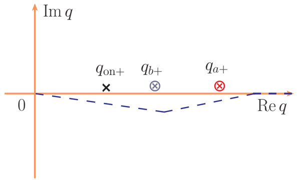
All singularities of the integrand of I:

$$q_{\text{on}+}, \qquad q_{a+} = \gamma\,(\beta\,E_2^* + p_2^*) + i\,\epsilon, \qquad q_{a-} = \gamma\,(\beta\,E_2^* - p_2^*) - i\,\epsilon,$$

$$q_{\text{on}-} < 0, \quad q_{b-} = -q_{a+} < 0 \text{ (for } \epsilon = 0), \qquad q_{b+} = -q_{a-},$$



(a) (b) (c)



2-body threshold singularity at

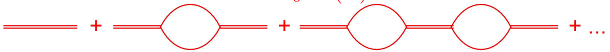
$m_{23} = m_2 + m_3$

triangle singularity at

$\boxed{p_{2,\text{left}} = p_{2,\text{right}}}$

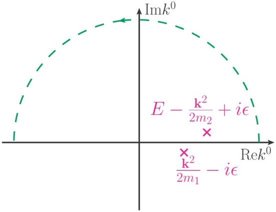
here $p_{2,\text{left}} = q_{\text{on}+}$, $p_{2,\text{right}} = q_{a-}$

We may also start from a QFT (for very small E_B, nonrelativistic)

Free field	Interacting field
$\frac{i}{E-E_0}$	$\frac{i}{E-E_0-\Sigma(E)}$
	

Here $E_0 = M_0 - m_1 - m_2$ with M_0 the bare mass, $\Sigma(E)$ is the self-energy (g_0: bare coupling constant)

$$\Sigma(E) = i g_0^2 \int \frac{d^4k}{(2\pi)^4} \left[\left(k^0 - \frac{\boldsymbol{k}^2}{2m_1} + i\epsilon \right) \left(E - k^0 - \frac{\boldsymbol{k}^2}{2m_2} + i\epsilon \right) \right]^{-1}$$



$$= -i2\mu g_0^2 (2\pi i) \int^{\Lambda} \frac{d^3\boldsymbol{k}}{(2\pi)^4} \frac{1}{2\mu E - \boldsymbol{k}^2 + i\epsilon}$$

$$= g_0^2 \frac{\mu}{2\pi} \sqrt{-2\mu E - i\epsilon} + \text{constant}$$

$$= g_0^2 \frac{\mu}{2\pi} \left[\sqrt{-2\mu E}\,\theta(-E) - i\sqrt{2\mu E}\,\theta(E) \right] + \text{constant}$$

The physical mass is $M = m_1 + m_2 + E_B (E_B \geq 0)$ with $-E_B$ the solution of $E - E_0 - \Sigma(E) = 0$, i.e.

$$\boxed{E_B = -E_0 - \Sigma(-E_B)}$$

Expanding the self-energy around the pole, we rewrite the propagator

$$\begin{aligned}\frac{i}{E - E_0 - \Sigma(E)} &= \frac{i}{E - E_0 - \left[\Sigma(-E_B) + (E + E_B)\Sigma'(-E_B) + \tilde{\Sigma}(E)\right]} \\ &= \frac{i}{E + E_B - (E + E_B)\Sigma'(-E_B) - \tilde{\Sigma}(E)} \\ &= \frac{iZ}{E + E_B - Z\tilde{\Sigma}(E)}\end{aligned}$$

Z is the wave function renormalization constant

$$\boxed{Z = \frac{1}{1 - \Sigma'(-E_B)} = \left[1 + \frac{g_0^2 \mu^2}{2\pi\sqrt{2\mu E_B}}\right]^{-1}}$$

The physical coupling constant

$$\tilde{g}^2 = Z g_0^2 = \frac{1}{\frac{1}{g_0^2} + \frac{\mu^2}{2\pi\sqrt{2\mu E_B}}} = (1-Z)\frac{2\pi}{\mu^2}\sqrt{2\mu E_B}$$

Taking into account the nonrel. normalization, we get the one in rel. QFT

$$\boxed{g^2 = 8 m_1 m_2 (m_1+m_2)\tilde{g}^2 = 16\pi(1-Z)(m_1+m_2)^2\sqrt{\frac{2E_B}{\mu}}}$$

If the ERE is dominated by the scattering length (when the pole is extremely close to threshold),

$$T(E) = \frac{2\pi/\mu}{-1/a - \sqrt{-2\mu E - i\epsilon}}$$

At LO, effective coupling strength for bound state

$$|g_{\rm NR}|^2 = \lim_{E\to -E_B} (E+E_B)\, T(E) = -\frac{2\pi}{\mu}\left(\frac{d}{dE}\sqrt{-2\mu E - i\epsilon}\right)^{-1}_{E=-E_B}$$

$$= \frac{2\pi}{\mu^2}\sqrt{2\mu E_B} \quad \Rightarrow \quad Z = 0 \text{ at this leading order approximation}$$